NATIONAL BUREAU OF STANDARDS REPORT

6998

MEAN MOTIONS IN CONDITIONALLY PERIODIC

SEPARABLE SYSTEMS

by

John P. Vinti

PROPERTY OF
SOUTHWEST RESEARCH INSTITUTE LIBRARY
SAN ANTONIO, TEXAS

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. Research projects are also performed for other government agencies when the work relates to and supplements the basic program of the Bureau or when the Bureau's unique competence is required. The scope of activities is suggested by the listing of divisions and sections on the inside of the back cover.

Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers. These papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three periodicals available from the Government Printing Office: The Journal of Research, published in four separate sections, presents complete scientific and technical papers; the Technical News Bulletin presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: Monographs, Applied Mathematics Series, Handbooks, Miscellaneous Publications, and Technical Notes.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.25) and its Supplement ($1.50), available from the Superintendent of Documents, Government Printing Office, Washington 25, D.C.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT **NBS REPORT**

1104-12-11440 November 10, 1960 6998

AFOSR TN 60-1254

MEAN MOTIONS IN CONDITIONALLY PERIODIC SEPARABLE SYSTEMS*

by

John P. Vinti

Mathematical Physics Section

*This work was supported by the U. S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

IMPORTANT NOTICE

NATIONAL BUREAU OF STAN intended for use within the Go to additional evaluation and revi listing of this Report, either in the Office of the Director, Natio however, by the Government ag to reproduce additional copies

Approved for public release by the Director of the National Institute of Standards and Technology (NIST) on October 9, 2015.

ress accounting documents ly published it is subjected oduction, or open-literature is obtained in writing from h permission is not needed, pared if that agency wishes

U. S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

MEAN MOTIONS IN CONDITIONALLY PERIODIC SEPARABLE SYSTEMS[1]

by

John P. Vinti

A search of the literature failed to disclose any general statement or proof of a theorem informally current among dynamical astronomers. The present paper gives a proof of the theorem, which states that, in any conditionally-periodic separable system, the mean frequency n_k of any separation coordinate q_k is equal to $\nu_k \equiv \partial\alpha_1/\partial J_k$. Here α_1 is the energy and J_k is the k'th action variable. The proof is carried out for non-singular Staeckel systems, so that it is applicable to any non-polar orbit of an artificial satellite, when the potential leads to separability.

1. Introduction

Conditionally periodic separable systems are commonly illustrated in works on advanced dynamics by the motion of a particle under the joint action of harmonic oscillator forces at right angles. This happens to be a very special case, for which each rectangular coordinate q_k has a constant frequency, equal to the corresponding "fundamental frequency" $\nu_k \equiv \partial\alpha_1/\partial J_k$, where α_1 is the energy and J_k the corresponding action variable.

In a more general system of this type, each generalized coordinate q_k may have a variable frequency, but it appears to be generally believed among dynamical astronomers that the mean frequency of q_k must be equal to ν_k, if the conditionally periodic system is separable. Needing to refer to such a theorem in solving a specific problem, I have searched the literature but have found no explicit statement or proof of it. The present paper is an attempt to furnish such a reference, with a proof sufficiently general to be applicable to all the separable problems that

1. This work was supported by the U. S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

may arise in the gravitational theory of the orbit of a satellite of an oblate planet.

Little of the analysis in this paper can claim to be really new. Much of the pertinent material in the literature, however, is discursive, relatively unavailable, and expressed in notations now unfamiliar to most mathematical physicists. Some of it is inadequate, if not incorrect, especially in the treatment of the periodicity of the q's as functions of the angle variables. Moreover, none of it seems to have been carried out in the shortest and most appropriate way to prove the theorem in question. The present paper attempts to give a concise and correct treatment that will serve this purpose.

It is easy to see why such a theorem should have escaped formal statement and proof. Physicists have not been concerned with mean frequencies of this kind. Dynamical astronomers have been, but ordinarily for non-separable systems. Until 1957 their only separable problems were the Kepler problem [2][2] for which the coordinate frequencies are all equal to the constant value $\nu_1 = \nu_2 = \nu_3$, and the problem of two centers, which has remained a curiosity to be found in Charlier's famous book [1], but without application. Since 1957 various potentials have been suggested, by Sterne [3], by Garfinkel [4], and by the author [5], for the gravitational field of an oblate planet, all of which lead to separability and to intermediary satellite orbits with $\nu_1 \neq \nu_2 \neq \nu_3$. Since the solution for these orbits is greatly facilitated by knowledge of the mean coordinate frequencies, it now becames desirable to have a formal and general proof of the theorem.

---

2. Figures in brackets indicate the literature references at the end of the paper.

It is now convenient to discuss briefly the general plan of the paper, without definitions. The difficulty in proving the theorem arises only when the fundamental frequencies $\nu_1 \ldots \nu_n$ are incommensurable. If $w_1 \ldots w_n$ are the angle variables, the plan is first to show that for a conditionally periodic system there exist infinitely many values of the time, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points separated from the initial point $w_1(0) \ldots w_n(0)$ by integer intervals $\Delta w_k = m_k$, $k = 1 \ldots n$. This fact follows directly from a theorem of Dirichlet, which is easy to understand and to apply.

To convert this result to q-space, it is necessary to know that q_k is a single-valued, continuous, periodic function of the w's. To show this the author restricts considerations to non-singular Staeckel systems, proving that for them each q_k is a single-valued differentiable function of $v_k \equiv \int dq_k/p_k$ and each v_k of the angle variables $w_1 \ldots w_n$. With a careful use of the single-valuedness, the periodic property then follows.

Application of these properties of the q's as functions of the w's shows that, at the values of the time mentioned above, the orbit in q-space then passes arbitrarily closely to points where each q_k would have gone through exactly m_k cycles. The proof of the theorem then follows.

2. Staeckel Systems.

If $q_1 \ldots q_n$ and $p_1 \ldots p_n$ denote the generalized coordinates and momenta of a dynamical system of n degrees of freedom, the system is said to be of the Staeckel type [1, 6, 7, 8, 9] if the Hamiltonian H is given by

$$H = \tfrac{1}{2} \sum_{k=1}^{n} A_k(q_1 \ldots q_n) p_k^2 + V(q_1 \ldots q_n), \quad A_k > 0, \quad (k = 1 \ldots n) \tag{1}$$

and if there exist functions $\phi_{ij}(q_i)$, $\psi_i(q_i)$, $i, j = 1 \ldots n$, such that

$$A_k = M_{kl} / \det(\phi_{ij}) \tag{2}$$

$$V = \sum_{k=1}^{n} \psi_k(q_k) A_k, \tag{3}$$

M_{kl} being the cofactor of $\phi_{kl}(q_k)$ in the determinant $\det(\phi_{ij})$.

Conditions (2) and (3) are necessary and sufficient for the separability of a system with such a Hamiltonian.

If we next define the domain Q of the q's as the totality of real values of $q_1 \ldots q_n$ for which $p_k^2 \geqq 0$, $k = 1 \ldots n$, we may then define a non-singular Staeckel system as one for which $\psi_k(q_k)$ and $\phi_{ki}(q_k)$, i, $k = 1 \ldots n$, exist and are single-valued and for which $\det(\phi_{ij}) \neq 0$ anywhere in Q. If we put

$$\Phi \equiv (\phi_{ij}(q_i)) \tag{4}$$

for the Staeckel matrix, then Φ^{-1} exists and is single-valued anywhere in Q; in particular

$$(\Phi^{-1})_{1k} = A_k \quad (k = 1 \ldots n) \tag{5}$$

all exist anywhere in Q. (This restriction thus rules out polar orbits from consideration if the right ascension ϕ is one of the coordinates, since A_3 then becomes infinite on the polar axis.)

The momenta p_k are then given by

$$p_k^2 = -2\psi_k(q_k) + 2 \sum_{i=1}^{n} \phi_{ki}(q_i)\alpha_i, \quad (k = 1 \ldots n) \tag{6}$$

where the α's are separation constants, α_1 being the energy. (For satellite problems, where $n = 3$, $2\alpha_2$ and $2\alpha_3$ are usually denoted by α_2^2 and α_3^2.)

The Hamilton-Jacobi function W is then given by

$$W = \sum_{k=1}^{n} \int_{q_{ko}}^{q_k} p_k dq_k = \sum_{k=1}^{n} (\pm) \int_{q_{ko}}^{q_k} [-2\psi_k(q_k) + 2 \sum_{i=1}^{n} \phi_{ki}(q_i)\alpha_i]^{\frac{1}{2}} dq_k, \tag{7}$$

where the sign is $\pm$ respectively as $dq_k \gtrless 0$.

3. Conditionally Periodic Staeckel Systems.

We call a Staeckel system conditionally periodic if each coordinate is either rotational or librational. A coordinate q_k is rotational if: (i) it is an angle, (ii) with $p_k^2 \equiv F_k(q_k)$ there exist positive real numbers c_{1k} and c_{2k} such that $c_{2k} \geq F_k(q_k) \geq c_{1k} > 0$ for all real values of q_k, and (iii) if

$$F_k(q_k + 2\pi) = F_k(q_k). \tag{8}$$

Note that $c_{1k} > 0$ rules out asymptotic motions and that the periodicity implied by (8) may reduce to simple constancy. The latter holds, e.g., when q_k is the right ascension ϕ of an artificial satellite, since there does not exist any potential, depending on the right ascension ϕ, which both leads to separability and remains finite on the polar axis.[3] For such a rotational coordinate q_k either $p_k > c_{1k}^{\frac{1}{2}}$ for all q_k or $p_k < -c_{1k}^{\frac{1}{2}}$ for all q_k. In either case

$$v_k(q_k) \equiv \int_{q_{ko}}^{q_k} dq_k/p_k \tag{9}$$

is a single-valued function of q_k, with derivative dv_k/dq_k existing and differing from zero for all values of q_k. Thus q_k is a single-valued differentiable[4] function of v_k.

---

3. See the tables in [9], pp. 656, 658, 660

4. Hereafter abbreviated to "s.v.d."

A coordinate q_i is librational if there exist real numbers a_i, b_i, C_{1i}, and C_{2i} and a real function $G_i(q_i)$ such that

$$p_i^2 = (q_i - a_i)(b_i - q_i)G_i(q_i), \tag{10.1}$$

with

$$C_{2i} \geq G_i(q_i) \geq C_{1i} > 0 \quad (a_i \leq q_i \leq b_i) \tag{10.2}$$

and

$$a_i \leq q_i(0) \leq b_i, \tag{10.3}$$

$q_i(0)$ being the initial value of q_i. If we again define v_i by (9), then

$$v_i(q_i) = \pm \int_{q_{io}}^{q_i} [(q_i - a_i)(b_i - q_i)G_i(q_i)]^{-\frac{1}{2}} dq_i, \tag{11}$$

where the sign is $\pm$ accordingly as $dq_i \gtrless 0$, respectively. Then

$$v_i = \int_{E_{io}}^{E_i} G_i^{-\frac{1}{2}}\, dE_i, \tag{12}$$

where the uniformizing variable E_i is defined by the equation

$$2q_i = a_i + b_i + (a_i - b_i)\cos E_i \tag{13}$$

and the requirement that E_i shall always increase as q_i varies. By (12) and (10.2) v_i is then a single-valued function of E_i, with derivative existing and non-vanishing for all E_i, so that E_i must be a s.v.d. function of v_i. By (13), however, q_i is a s.v.d. function of E_i, so that, finally, q_i is a s.v.d. function of v_i. Thus in a conditionally periodic Staeckel system any coordinate q_k is a s.v.d. function of the corresponding v_k.

4. The v_k's as Functions of the Angle-Variables w_k.

If we now let an increase of 2π be one cycle of a rotational coordinate and a single round trip from a_k to b_k be one cycle of a librational coordinate, we may define the action and angle variables J_k and w_k by

$$J_k \equiv \oint p_k dq_k \tag{14}$$

and

$$w_k \equiv \partial W/\partial J_k, \tag{15}$$

where W is now to be considered a function of the q's and the J's, rather than of the q's and the α's. It is well known [10, 11] that J_k and w_k are canonically conjugate and that

$$\dot{w}_k = \partial\alpha_1/\partial J_k = \nu_k . \tag{16}$$

If we also define the Jacobi variables B_i by

$$B_i \equiv \partial W/\partial\alpha_i, \tag{17}$$

we obtain

$$B_i = \sum_{k=1}^{n} \frac{\partial W}{\partial J_k} \frac{\partial J_k}{\partial \alpha} = \sum_{k=1}^{n} w_k \omega_{ki}, \tag{18}$$

where

$$\omega_{ki} = \partial J_k/\partial\alpha_i \tag{19}$$

Increments $dw_1 \ldots dw_n$ then lead to

$$dB_i = \sum_{k=1}^{n} (dw_k)\,\omega_{ki} \qquad (i = 1 \ldots n). \tag{20}$$

But, by (7) and (17)

$$dB_i = \sum_{k=1}^{n} \frac{\partial p_k}{\partial \alpha_i}\, dq_k = \tfrac{1}{2} \sum_{k=1}^{n} \frac{\partial p_k^2}{\partial \alpha_i}\, dq_k/p_k \tag{21}$$

or

$$dB_i = \sum_{k=1}^{n} \phi_{ki}(q_k) dv_k, \tag{22}$$

by (21), (6), and (9). Also, by (19), (14), (6), and (9)

$$\omega_{ki} = \oint \phi_{ki}(q_i) dv_k \,. \tag{23}$$

If we now introduce the matrix Φ, the matrix $\Omega \equiv (\omega_{ki})$, and the row matrices $dv \equiv (dv_1, \ldots dv_n)$ and $dw = (dw_1, \ldots dw_n)$, we find from (20) and (22)

$$dv\,\Phi = dw\,\Omega. \tag{24}$$

For a non-singular Staeckel system[5] Φ, Φ^{-1}, and Ω all exist at every point of Q, so that

$$dv = dw\,\Omega\,\Phi^{-1} \tag{25}$$

or

$$\partial v_i/\partial w_k = (\Omega\,\Phi^{-1})_{ki}. \tag{26}$$

Thus each derivative $\partial v_i/\partial w_k$ exists and is single-valued everywhere in Q. Now the J's are all real, by (14). Thus, by (7) and (15), if the p's are all real, then W and the w's are all real; if some of the p's are non-real, then W is non-real and so are some of the w's. If all the w's are real, it then follows that all the p's are real, else we should have a contradiction. Thus the domain Q, corresponding to the totality of all real values of the q's for which the p's are all real, also corresponds exactly to the set of all possible real values for all the w's. It therefore follows that each derivative $\partial v_i/\partial w_k$ exists and is single-valued at any point in w-space. Thus each v_k must be a s.v.d. function of $w_1 \ldots w_n$. In Section 3, however, we showed that each q_k is a s.v.d. function of the corresponding v_k. Thus for a conditionally periodic non-singular Staeckel system each q_k is a s.v.d. function $f_k(w_1 \ldots w_n)$.

5. The Periodic Properties of $q_k = f_k(w_1 \ldots w_n)$.

By (7) and (15)

$$dw_k = \sum_{i=1}^{n} \frac{\partial p_i}{\partial J_k}\, dq_i. \tag{27}$$

---

5. See Appendix for examples.

If now each coordinate q_i goes through an integral number m_i of cycles, then by a familiar argument

$$\Delta w_k = \sum_{i=1}^{n} m_i \oint (\partial p_i/\partial J_k) dq_i = \frac{\partial}{\partial J_k} \sum_{i=1}^{n} m_i \oint p_i dq_i = m_k \ . \quad (28)$$

Thus if each q_k goes through exactly m_k cycles, each w_k increases by the integer m_k. (Note, however, that such simultaneous increases are not always physically possible: this section is thus concerned only with the mathematical properties of the functions $f_k(w_1 \ldots w_n)$.)

But we are really interested in the inverse problem where each w_k has increased by an integer m_k and we ask what has happened to the q's. Now the q's are uniquely determined by the w's, because of the single-valued property. In the situation of the preceding paragraph where each librational coordinate returns to its initial value and each rotational coordinate q_i increases by $2\pi m_i$, each angle variable w_k increases by m_k. Since the w's determine the q's uniquely, this has the result that whenever $\Delta w_k = m_k$, $k = 1 \ldots n$, each librational coordinate returns to its initial value and each rotational coordinate q_i increases by $2\pi m_i$.

Thus, in the inverse problem, whenever we are given $\Delta w_k = m_k$, $k = 1 \ldots n$, we find that each rotational coordinate q_i must go through exactly m_i cycles and that each librational coordinate q_j must go through some integral number of cycles, τ_j, say. By (28), however, we then find $\tau_j = m_j$. Thus whenever the angle variables w_k are all increased by integer amounts $\Delta w_k = m_k$, each of the functions $q_k = f_k(w_1 \ldots w_n)$ must go through exactly m_k cycles.

6. Mean Motions.

If in a time interval T the number of complete cycles passed through by any coordinate q_k is N_k, the corresponding mean frequency n_k is, by definition

$$n_k = \lim_{T \to \infty} (N_k/T), \quad (29)$$

if the limit exists. We shall now prove that $n_k = \nu_k = \partial\alpha_1/\partial J_k$, $k = 1 \ldots n$, for any conditionally periodic non-singular Staeckel system.

To do so, note that if $\nu_1 \ldots \nu_n$ are all commensurable, there exist a positive ν_o and positive integers $m_1 \ldots m_n$ such that

$$\nu_k = m_k \nu_o, \quad (k = 1 \ldots n) \tag{30}$$

where we may choose ν_o to be the greatest common divisor of the ν_k's. Then, from (16) and (30), during the actual motion,

$$w_k = w_k(0) + m_k \nu_o t \quad (k = 1 \ldots n) \tag{31}$$

and in the time interval $T \equiv 1/\nu_o$ we have

$$\Delta w_k = m_k \quad (k = 1 \ldots n) \tag{32}$$

By Section (5) each q_k goes through exactly m_k cycles in this time, so that in this case the motion is truly periodic, with period $1/\nu_o$. The mean frequency of q_k is thus

$$n_k \equiv m_k/T = m_k \nu_o = \nu_k \,. \tag{33}$$

If the frequencies $\nu_1 \ldots \nu_n$ are not all commensurable, we may let

$$\xi_k \equiv \nu_k/\nu_1 \quad (k = 1 \ldots n) \tag{34}$$

and then at least one of the ξ's will be irrational. Then by (16) and (34), during the actual motion

$$w_k = w_k(0) + \xi_k \nu_1 t \,. \tag{35}$$

We now use a theorem of Dirichlet [12], which states that if the set of real numbers $\xi_1 \ldots \xi_n$ has at least one irrational member, then the system of inequalities

$$|\xi_k - m_k/P| < P^{-1-\frac{1}{n}} \quad (k = 1 \ldots n) \tag{36}$$

has an infinite number of integer solutions for P and the m's. Note that the solutions for P have no upper bound.

To apply this theorem, consider only those values of the time interval T such that $\nu_1 T = P$, where P is an integer that satisfies (36). In this time each w_k increases from its initial value $w_k(0)$ to a final value given by

$$w_k(T) = w_k(0) + P\xi_k, \tag{37}$$

by (35). But by (36)

$$P\xi_k = m_k + \eta_k, \qquad |\eta_k| < P^{-\frac{1}{n}} \tag{38}$$

so that

$$w_k(T) = w_k(0) + m_k + \eta_k. \tag{39}$$

As $\nu_1 T = P$ takes on those larger and larger integer values corresponding to solutions of (36), each η_k approaches zero, by (38). Then, by (39), there exist infinitely many values of T, with no upper bound, at which the orbit in w-space passes arbitrarily closely to points where $\Delta w_k = m_k$, $k = 1 \ldots n$, the m_k's being solutions of (36).

If the initial q's are given by

$$q_k(0) = f_k[w_1(0), \ldots w_n(0)], \qquad (k = 1 \ldots n) \tag{40}$$

then the values of the q's at any of these times T are given by

$$q_k(T) = f_k[w_1(0) + m_1 + \eta_1, \ldots w_n(0) + m_n + \eta_n] \qquad (k = 1 \ldots n). \tag{41}$$

As we let $T = P/\nu_1$ assume those larger and larger values already referred to, the q's then approach arbitrarily closely to the values

$$q_k^*(T) = f_k[w_1(0) + m_1, \ldots w_n(0) + m_n] \qquad (k = 1 \ldots n). \tag{42}$$

This conclusion follows from (38) and the single-valuedness and differentiability of the functions f_k.

Comparison of (40) and (42) then shows that the values $q_k^*(T)$ correspond to $\Delta w_k = m_k$, $k = 1 \ldots n$, and are thus, by Section 5, the values that would be reached after each q_k had gone through exactly m_k cycles. Now, by the definition (29), it follows that the mean frequency

$$n_k = \lim_{T\to\infty} (m_k/T), \tag{43}$$

if the limit exists. But $m_k/T = \nu_1 m_k/P$ and, as $T\to\infty$, $\lim(m_k/P) = \xi_k$, by (36). Thus

$$n_k = \nu_1 \xi_k = \nu_k \tag{44}$$

by (43) and (34).

Thus, for each coordinate q_k of a conditionally periodic non-singular Staeckel system, the mean frequency n_k is equal to the corresponding fundamental frequency $\nu_k \equiv \partial\alpha_1/\partial J_k$.

APPENDIX

For theories of satellite orbits, appropriate coordinates are spherical or oblate spheroidal. The corresponding Staeckel matrices and their inverses are, if $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$:

Spherical

$$\Phi = \begin{pmatrix} 1 & -r^{-2} & 0 \\ 0 & 1 & -\csc^2\theta \\ 0 & 0 & 1 \end{pmatrix} \qquad \Phi^{-1} = \begin{pmatrix} 1 & r^{-2} & r^{-2}\csc^2\theta \\ 0 & 1 & \csc^2\theta \\ 0 & 0 & 1 \end{pmatrix}$$

or, if $x = c[(\xi^2+1)(1-\eta^2)]^{\frac{1}{2}}\cos\phi$, $y = c[(\xi^2+1)(1-\eta^2]^{\frac{1}{2}}\sin\phi$, $z = c\xi\eta$:

Oblate Spheroidal

$$\Phi = \begin{pmatrix} c^2\xi^2(\xi^2+1)^{-1} & -(\xi^2+1)^{-1} & (\xi^2+1)^{-2} \\ c^2\eta^2(1-\eta^2)^{-1} & (1-\eta^2)^{-1} & -(1-\eta^2)^{-2} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\Phi^{-1} = \begin{pmatrix} c^{-2}(\xi^2+1)(\xi^2+\eta^2)^{-1} & c^{-2}(1-\eta^2)(\xi^2+\eta^2)^{-1} & c^{-2}(\xi^2+1)^{-1}(1-\eta^2)^{-1} \\ -\eta^2(\xi^2+1)(\xi^2+\eta^2)^{-1} & \xi^2(1-\eta^2)(\xi^2+\eta^2)^{-1} & (\xi^2+\eta^2)^{-1}[\xi^2(1-\eta^2)^{-1} \\ & & +\eta^2(1+\xi^2)^{-1}] \\ 0 & 0 & 1 \end{pmatrix}.$$

For spherical coordinates Φ is most easily written down from the expressions for p_r^2, p_θ^2, and p_ϕ^2 in the Kepler problem [2], with replacement of α_2^2 and α_3^2 by $2\alpha_2$ and $2\alpha_3$. For oblate spheroidal coordinates Φ may be found by comparing equations (53) and (59.1) of [5] with Eq. (6) of the present paper.

Note that Φ or $\bar{\Phi}^{-1}$ could fail to exist only when $\sin\theta = 0$ or when $\eta^2 = 1$. This could happen only when the satellite goes over a pole and thus only in a polar orbit. Such a singularity in a polar orbit, however, is to be expected, since non-singularity of a Staeckel system leads to the q's being differentiable functions of the w's and thus of the time. In a polar orbit, on the other hand, the right ascension $q_3 \equiv \phi$ is a discontinuous function of time, being constant except at polar crossings, where it changes by π.

REFERENCES

1. C. L. Charlier, "Die Mechanik des Himmels", Veit and Comp., Leipzig 1902, Vol. I.
2. W. M. Smart, "Celestial Mechanics", Longmans, Green, & Co., New York 1953, p. 144.
3. T. E. Sterne, Astron. J. 62, 96(1957); 63, 28-40(1958).
4. B. Garfinkel, Astron. J. 63, 88-96(1958); 64, 353-367(1959).
5. J. P. Vinti, Phys. Rev. Letters 3, 8(1959); J. Research National Bureau of Standards 63B, 105-116(1959).
6. P. Staeckel, Habilitationsschrift, Halle 1891.
7. L. P. Eisenhart, Annals of Math. 35, 284-305(1934).
8. L. P. Eisenhart, Phys. Rev. 74, 87-89(1948).
9. P. M. Morse and H. Feshbach, "Methods of Theoretical Physics", McGraw-Hill, New York, N.Y. 1953, Vol. I, pp. 509-518, pp. 655-664.
10. P. Frank and R. v. Mises, "Die Differential-und Integralgleichungen der Physik", Vieweg & Sohn, Braunschweig 1935, Vol. II, pp. 94-101.
11. Max Born, "The Mechanics of the Atom", G. Bell and Sons, Ltd., London 1927, Chapters 1 and 2.
12. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers", Clarendon Press, Oxford 1938, p. 169, Theorem 200.

USCOMM-NBS-DC

U.S. DEPARTMENT OF COMMERCE
Frederick H. Mueller, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards at its major laboratories in Washington, D.C., and Boulder, Colo., is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section carries out specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant publications, appears on the inside of the front cover.

WASHINGTON, D.C.

ELECTRICITY. Resistance and Reactance. Electrochemistry. Electrical Instruments. Magnetic Measurements. Dielectrics.

METROLOGY. Photometry and Colorimetry. Refractometry. Photographic Research. Length. Engineering Metrology. Mass and Scale. Volumetry and Densimetry.

HEAT. Temperature Physics. Heat Measurements. Cryogenic Physics. Rheology. Molecular Kinetics. Free Radicals Research. Equation of State. Statistical Physics. Molecular Spectroscopy.

RADIATION PHYSICS. X-Ray. Radioactivity. Radiation Theory. High Energy Radiation. Radiological Equipment. Nucleonic Instrumentation. Neutron Physics.

CHEMISTRY. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Molecular Structure and Properties of Gases. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

MECHANICS. Sound. Pressure and Vacuum. Fluid Mechanics. Engineering Mechanics. Combustion Controls.

ORGANIC AND FIBROUS MATERIALS. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Plastics. Dental Research.

METALLURGY. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion. Metal Physics.

MINERAL PRODUCTS. Engineering Ceramics. Glass. Refractories. Enameled Metals. Constitution and Microstructure.

BUILDING RESEARCH. Structural Engineering. Fire Research. Mechanical Systems. Organic Building Materials. Codes and Safety Standards. Heat Transfer. Inorganic Building Materials.

APPLIED MATHEMATICS. Numerical Analysis. Computation. Statistical Engineering. Mathematical Physics.

DATA PROCESSING SYSTEMS. Components and Techniques. Digital Circuitry. Digital Systems. Analog Systems. Applications Engineering.

ATOMIC PHYSICS. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics.

INSTRUMENTATION. Engineering Electronics. Electron Devices. Electronic Instrumentation. Mechanical Instruments. Basic Instrumentation.

Office of Weights and Measures.

BOULDER, COLO.

CRYOGENIC ENGINEERING. Cryogenic Equipment. Cryogenic Processes. Properties of Materials. Gas Liquefaction.

IONOSPHERE RESEARCH AND PROPAGATION. Low Frequency and Very Low Frequency Research. Ionosphere Research. Prediction Services. Sun-Earth Relationships. Field Engineering. Radio Warning Services.

RADIO PROPAGATION ENGINEERING. Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation-Terrain Effects. Radio-Meteorology. Lower Atmosphere Physics.

RADIO STANDARDS. High frequency Electrical Standards. Radio Broadcast Service. Radio and Microwave Materials. Atomic Frequency and Time Standards. Electronic Calibration Center. Millimeter-Wave Research. Microwave Circuit Standards.

RADIO SYSTEMS. High Frequency and Very High Frequency Research. Modulation Research. Antenna Research. Navigation Systems. Space Telecommunications.

UPPER ATMOSPHERE AND SPACE PHYSICS. Upper Atmosphere and Plasma Physics. Ionosphere and Exosphere Scatter. Airglow and Aurora. Ionospheric Radio Astronomy.

NBS